Subwavelength localisation in aperiodic block resonator systems

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Joint work with Habib Ammari, Silvio Barandun, Bryn Davies, Erik Hiltunen and Clemens Thalhammer

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Motivation

- **Periodic systems:** Floquet-Bloch and Toeplitz theory $\implies$ Eigenvalues lie along bands and eigenmodes are quasiperiodic and delocalised.

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- **Electrons in disordered systems:** Disorder induces localisation [Anderson (1958) and Thouless (1974)], Band gaps in disordered systems [Saxon and Hutner (1949) and Hori and Matsuda (1964)]
- **Our interest:**
  - Is there a classical analogue?
  - Can we construct aperiodic disordered systems with predictable properties?

1. Setting
   - High-contrast subwavelength resonators
   - Block disordered systems
2. Guaranteed bandgaps using transfer matrices
   - Infinite block disordered systems
   - Relation to finite block disordered systems
3. Hybridisation regions and the meta-atom estimate
4. Conclusion

High-contrast subwavelength resonance problem

Setting: Time-harmonic wave resonance in a 1D array of N resonators $\mathcal{D} = \bigsqcup_{i=1}^{N} (x_i^{\mathrm{L}}, x_i^{\mathrm{R}})$ described by the coupled Helmholtz equations

$$
\begin{cases}
\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} u + \dfrac{\omega^2}{v^2} u = 0, \quad \text{in } \mathbb{R} \setminus \mathcal{D}, \\
\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} u + \dfrac{\omega^2}{v_i^2} u = 0, \quad \text{in } D_i \coloneqq (x_i^{\mathrm{L}}, x_i^{\mathrm{R}}), \\
u|_{\mathrm{R}}(x_i^{\mathrm{L,R}}) - u|_{\mathrm{L}}(x_i^{\mathrm{L,R}}) = 0, \\
\left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{\mathrm{in}} (x_i^{\mathrm{L,R}}) - \dfrac{\rho_{\mathrm{in}}}{\rho_{\mathrm{out}}} \left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{\mathrm{out}} (x_i^{\mathrm{L,R}}) = 0, \\
u \text{ satisfies the Sommerfeld radiation condition}
\end{cases}
\tag{1}
$$

Densities $\rho_{\mathrm{in}}, \rho_{\mathrm{out}}$ and wave speeds v_i, v
Resonator lengths $\ell_i = x_i^{\mathrm{R}} - x_i^{\mathrm{L}}$
Resonator spacings $s_i = x_{i+1}^{\mathrm{L}} - x_i^{\mathrm{R}}$
Contrast ratio $\delta \coloneqq \rho_{\mathrm{in}} / \rho_{\mathrm{out}}$

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Regime: Looking for Subwavelength resonant frequencies in the high-contrast regime, i.e. resonant frequencies ω with

$$\omega(\delta) \to 0 \quad \text{as} \quad \delta \to 0.$$

Capacitance matrix approximation

The subwavelength solutions are approximated in leading order by the eigenvalue problem of the generalised capacitance matrix VC

Theorem (From Feppon, Cheng, and Ammari (2023))

Equation (1) *has N subwavelength eigenfrequencies* ω_i*, given by*

$$\omega_i = \sqrt{\delta \lambda_i} + \mathcal{O}(\delta) \quad \text{as} \quad \delta \to 0 \tag{2}$$

where $(\lambda_i)_{1 \leq i \leq N}$ *are the eigenvalues of the eigenvalue problem*

$$VC\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i.$$

Furthermore, let u_i *be a subwavelength eigenmode corresponding to* ω_i *and let* $\boldsymbol{u}_i$ *be the corresponding eigenvector of* VC*. Then,* $u_i(x) = \boldsymbol{u}_i^{(j)} + \mathcal{O}(\delta)$ *for* $x \in D_j$ *where* $\boldsymbol{u}^{(j)}$ *denotes the* j*-th entry of the eigenvector.*

Material matrix V

- $V = \operatorname{diag}\left(\frac{v_1^2}{\ell_1}, \dots, \frac{v_N^2}{\ell_N}\right)$

Capacitance matrix C

- Solely determined by resonator spacings $(s_i)_{i=1}^{N-1}$
- Tridiagonal and symmetric

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Blocks: Library of $D \in \mathbb{N}$ blocks $B_1, \ldots, B_D$ each containing one or more resonators

Sequence: Arrange blocks according to a block sequence $\chi \in \{1, \ldots, D\}^M, \{1, \ldots, D\}^{\mathbb{Z}}$

- Block sequences are sampled i.i.d. with probabilities $p_1, \ldots, p_D$.

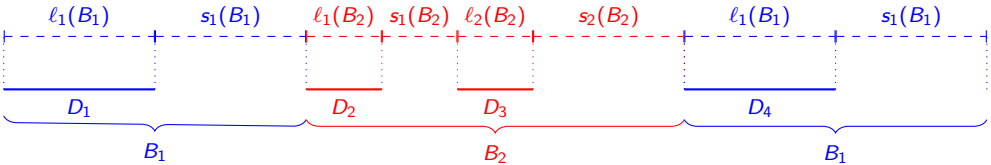


Figure: Schematic of a block disordered system built from B_1 (monomer) and B_2 (dimer) blocks, arranged according to the sequence $\chi = (1, 2, 1)$.

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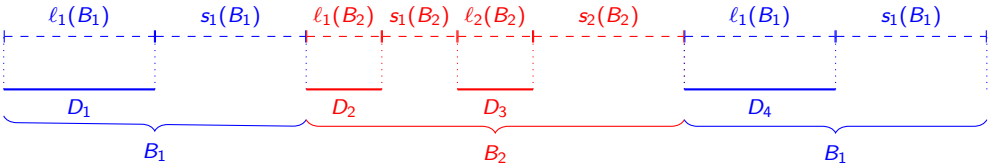


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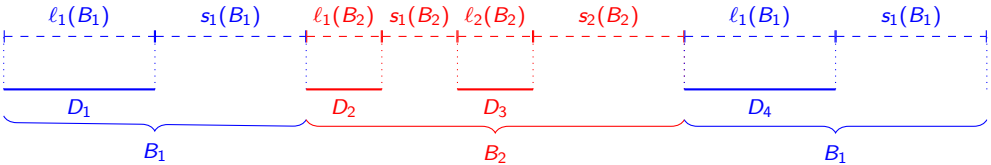
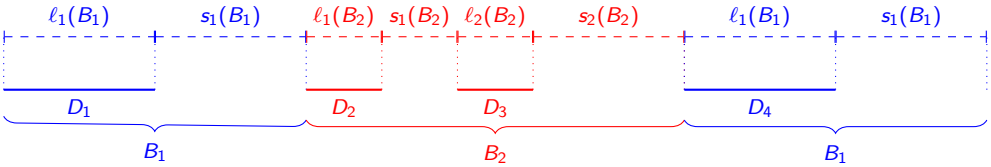
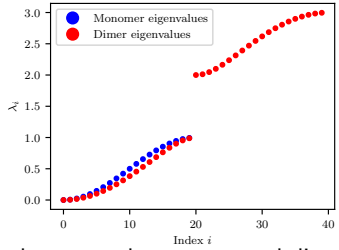


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Standard blocks





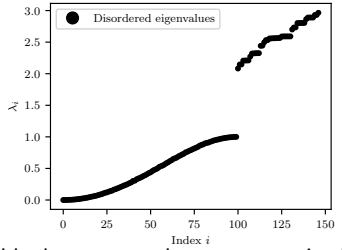
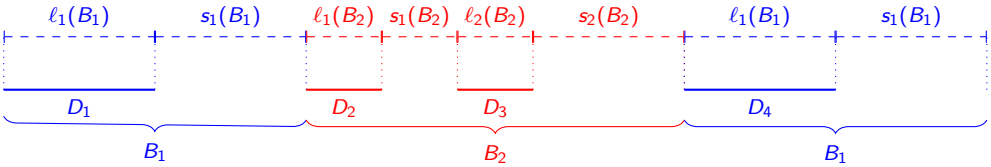


Figure: Spectra of finitely repeated monomer and dimer blocks compared to a composite block disordered system.

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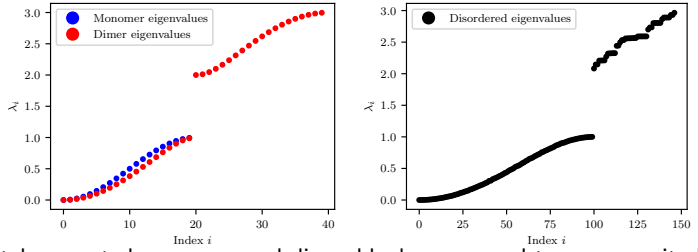


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How are the spectral properties of the disordered system related to its building blocks?

Jacobi operators and transfer matrix

Goal: Saxon-Hutner type result[1], *i.e.* characterise bandgaps of large block disordered systems in terms of their constituent block bandgaps

- As the block number $M \to \infty$, the matrices VC converge strongly to a Jacobi operator on $\ell^2(\mathbb{Z})$,

$$\begin{aligned} \mathcal{J} : \ell^2(\mathbb{Z}) &\to \ell^2(\mathbb{Z}) \\ \boldsymbol{u}^{(i)} &\mapsto (J\boldsymbol{u})^{(i)} = \boldsymbol{a}^{(i-1)}\boldsymbol{u}^{(i-1)} + \boldsymbol{a}^{(i)}\boldsymbol{u}^{(i+1)} + \boldsymbol{b}^{(i)}\boldsymbol{u}^{(i)}. \end{aligned}$$

with

$$\boldsymbol{a}^{(i)} = -\frac{v_i v_{i+1}}{s_i\sqrt{\ell_i \ell_{i+1}}} \quad \text{and} \quad \boldsymbol{b}^{(i)} = \frac{v_i^2}{\ell_i}\left(\frac{1}{s_{i-1}} + \frac{1}{s_i}\right). \tag{3}$$

[1] Hori and Matsuda, 1964.

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- We study the spectral equation

$$(\mathcal{J} - \lambda)\boldsymbol{u} = 0 \tag{4}$$

with symbolic solutions $\boldsymbol{u} \in \mathbb{C}^{\mathbb{Z}}$ (not necessarily in $\ell^2(\mathbb{Z})$ or even bounded)

- Define the transfer matrix cocycle as the matrix sequence $T^\lambda : \mathbb{Z} \to \mathrm{GL}(2,\mathbb{R})$ given by

$$T^\lambda(i) := \frac{1}{\boldsymbol{a}^{(i)}}\begin{pmatrix} \lambda - \boldsymbol{b}^{(i)} & -\boldsymbol{a}^{(i-1)} \\ \boldsymbol{a}^{(i)} & 0 \end{pmatrix} \quad \text{and have} \quad \begin{pmatrix} \boldsymbol{u}^{(i+1)} \\ \boldsymbol{u}^{(i)} \end{pmatrix} = T^\lambda(i)\begin{pmatrix} \boldsymbol{u}^{(i)} \\ \boldsymbol{u}^{(i-1)} \end{pmatrix}$$

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Uniform hyperbolicity and Johnson's theorem

- Correspondence $\mathrm{SL}(2,\mathbb{R}) \leftrightarrow$ Möbius transforms on $\mathbb{R}\mathbb{P}^1$
- **Uniform hyperbolicity:** A cocycle $P \in \ell_\infty(\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ is uniformly hyperbolic (UH) if there exist two sequences $s, u : \mathbb{Z} \to \mathbb{R}\mathbb{P}^1$ (called the stable and unstable direction) such that
  1. P-**invariance.** For every $i \in \mathbb{Z}$,
  $$P(i)u(i) = u(i+1) \quad \text{and} \quad P(i)s(i) = s(i+1).$$
  2. **Uniform growth / decay.** There exist $C > 0$ and $\eta > 1$ such that
  $$\|P_n(i)\vec{s}(i)\|, \|P_{-n}(i)\vec{u}(i)\| \leq C\eta^{-n}$$
  for all $i \in \mathbb{Z}$, $n \in \mathbb{N}$ and all unit vectors $\vec{u}(i) \in u(i)$ and $\vec{s}(i) \in s(i)$.

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- **Johnson's Theorem**[2]**:**
$$\lambda \notin \sigma(\mathcal{J}) \iff T^\lambda \text{ is uniformly hyperbolic}$$

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Propagation matrix

Issues:

1. The transfer matrix $T^{\lambda}(i)$ depends on the properties of the $i-1^{\text{th}}$, i^{th} and $i+1^{\text{th}}$ resonator
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Remedies:

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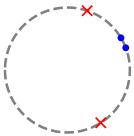
2. **Block sequence compression:** For every block B_d we define the block propagation matrix

$$\mathcal{P}_d^\lambda := \prod_{k=1}^{\mathrm{len}(B_d)} P^\lambda_{\ell_k(B_d), s_k(B_d), v_k(B_d)}$$

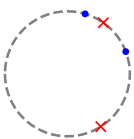
   and can define the block propagation matrix cocycle $\mathcal{P}^\lambda : \mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ as $\mathcal{P}^\lambda(j) := \mathcal{P}^\lambda_{\chi^{(j)}}$

   $\implies \mathcal{P}^\lambda(j)$ takes on finitely many distinct values which appear *i.i.d.*

Saxon-Hutner for infinite block disordered systems



(a) Source-sink condition satisfied



(b) Source-sink condition violated

- We sort the source and sink eigenpairs $(\xi_s, E_s), (\xi_u, E_u)$ of $\mathcal{P}_d^\lambda$ such that $|\xi_s| \leq 1 \leq |\xi_u|$ and have the correspondence $E_{s,u} \subset \mathbb{R}^2 \leftrightarrow s, u \in \mathbb{RP}^1$

Theorem

Let $\lambda \geq 0$ and assume that

1. *$\mathcal{P}_d^\lambda$ is hyperbolic (i.e. $\left|\xi_u(\mathcal{P}_d^\lambda)\right| > 1$) for every $d = 1, \ldots, D$,*
2. *all sink fixed points $u(\mathcal{P}_1^\lambda), \ldots, u(\mathcal{P}_D^\lambda)$ lie in the same connected component of $\mathbb{RP}^1 \setminus \{s(\mathcal{P}_1^\lambda), \ldots, s(\mathcal{P}_D^\lambda)\}$.*

Then the transfer matrix cocycle T^λ is uniformly hyperbolic and thus $\lambda \notin \sigma(\mathcal{J})$.

- **Proof:** Assumptions 1&2 $\implies$ family of Möbius transformations $\{P_d^\lambda\}$ has an invariant cone $\implies$ block propagation matrix cocycle is UH $\implies$ transfer matrix cocycle is UH
- Our standard monomer and dimer blocks satisfy the source sink condition for all λ

Edge modes and the finite setting

- **Goal:** Extend the Saxon-Hutner theorem to finite block disordered systems
- **Main idea:** Use $\sigma(VC) \subset \sigma(\mathcal{J}_+) \cup \sigma(\mathcal{J}_-)$ where $\mathcal{J}_\pm$ are semi-infinite Jacobi operators with physical boundary conditions

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- Left edge modes (*i.e.* edge modes of $\mathcal{J}_+$) occur iff $s(0)$ of $\mathcal{P}^\lambda$ exactly matches the boundary condition, *i.e.*

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- **Resolution:** Edge modes can be prevented with an additional source-sink condition on $(1, 0)^\top$

$\Longrightarrow$ Saxon-Hutner type characterisation holds in the finite case as well

Saxon-Hutner in action and tripartite density of states

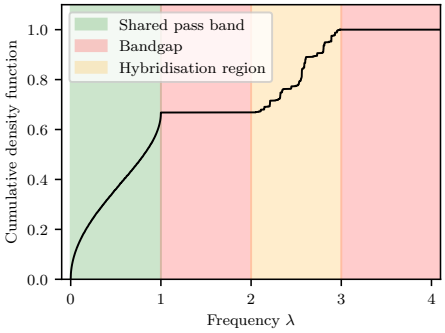


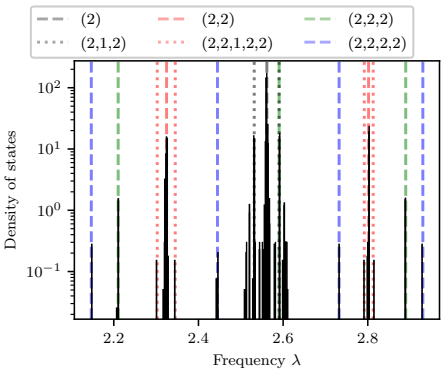
Figure: CDF of a block disordered monomer-dimer system overlaid atop the distinct spectral regions determined by the block bandgaps.

- Define the cumulative distribution function and the density of states as

$$F(A,\lambda) := \frac{1}{N}\left|\sigma(A) \cap (-\infty, \lambda]\right| \quad D(A,\lambda) := \frac{\mathrm{d}}{\mathrm{d}\lambda} F(A,\lambda)$$

- **Three distinct spectral regions:**
  - Bandgap (Only hyperbolic blocks) $\implies$ zero DoS
  - Shared pass band (No hyperbolic block) $\implies$ nonzero DoS, regular band
  - Hybridisation region (Some hyperbolic blocks) $\implies$ nonzero DoS, irregular band

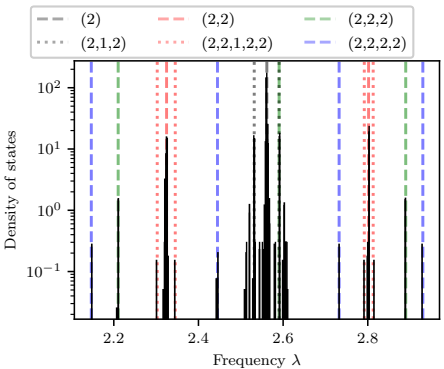
Weakly coupled defect modes



What is going on in the hybridisation region?

- **Figure:** DoS in the hybridisation region of the monomer-dimer block example, $[2, 3]$ lies in the gap of the monomer block but in the band of the dimer block.
- Local arrangements of pass band blocks (Meta-atoms) act as isolated defects and contribute resonant modes in the hybridisation region, leading to a fractal-like DoS.
- **Figure:** Vertical lines mark meta-atom defect modes. Closely match the DoS peaks with peak height proportional to the meta-atom probability

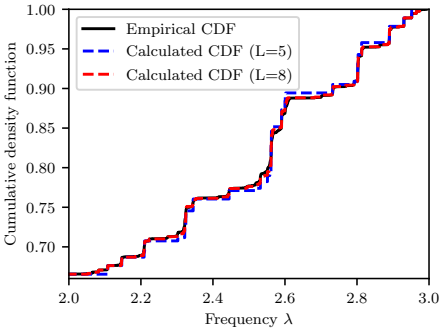
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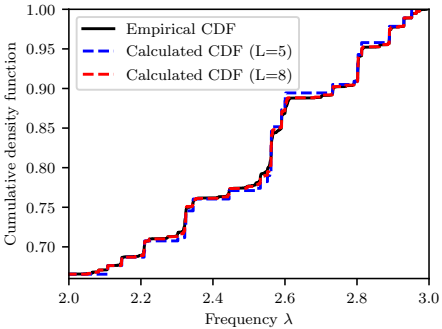
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- **Figure:** Vertical lines mark meta-atom defect modes. Closely match the DoS peaks with peak height proportional to the meta-atom probability
- DoS slightly smoothed by weak interactions between these modes (hybridisation)

Meta-atom estimate



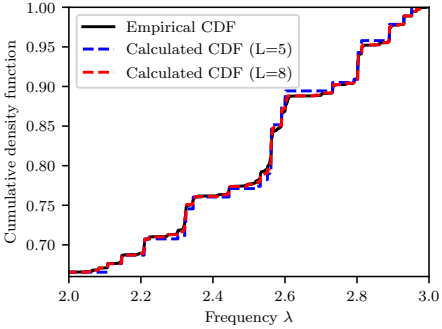
- **Goal:** Estimate the DoS in hybridisation regions.
- **Approach:** Leverage the fractal-like concentration of DoS around meta-atom modes induced by isolation
- **Key Steps:**
  1. Pre-compute meta-atom spectra when embedded in bandgap blocks (up to some maximal meta-atom length L).
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This approach continues to work for non-*i.i.d.* block sequences χ, *i.e.* dependent, quasiperiodic or hyperuniform

Conclusion

Recap:

1. Introduced block disordered systems as a tractable model for aperiodic media
2. Used transfer matrices to find guaranteed bandgaps from block properties
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Further directions:

- **Global localisation and Lyapunov exponent prediction:** Use transfer matrix framework to prove eigenmode localisation and estimate Lyapunov exponents
- **Non-reciprocal disordered systems:** Competition between localisation due to non-reciprocity and disorder; Lyapunov exponent splits into a non-reciprocal and disorder part
- **Higher dimensions:** Weaker Anderson localisation $\implies$ more delocalisation, long-range interaction, percolation-type phenomena in the hybridisation regions?

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Thank you for your attention!

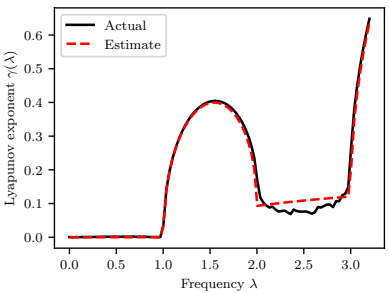
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Further results of the transfer matrix framework

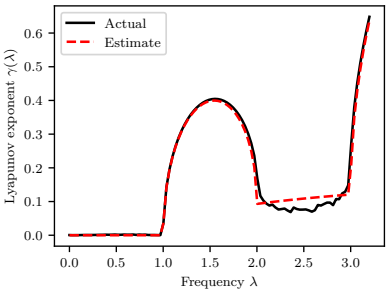


- Define the Lyapunov exponent of an infinite block disordered system as

$$\gamma(\lambda) := \lim_{N\to\infty} \frac{1}{N} \ln \left\| \prod_{i=1}^{N} T^{\lambda}(i) \right\|$$

- **Furstenberg's Theorem:** $\gamma(\lambda) > 0$ almost surely;
- **Kotani Theory:** $\gamma(\lambda) > 0 \implies$ no *a.c.* spectrum at λ
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- **Lyapunov estimate:** We can approximate the Lyapunov exponent as

$$\gamma(\lambda) \approx \frac{\sum_{d=1}^{D} p_d \ln \rho(\mathcal{P}_d^{\lambda})}{\sum_{d=1}^{D} p_d \operatorname{len}(B_d)} = \frac{\mathbb{E} \ln \rho(\mathcal{P}_d^{\lambda})}{\mathbb{E} \operatorname{len}(B_d)}$$

- This estimate relies on the sink eigenspaces of $\mathcal{P}_d^{\lambda}$ being somewhat parallel